Binomial Theorem

For $n \in \mathbb{N}$,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Ex: Show $9^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^{n-k}$

Since $9 = 10-1$, $9^n = (10-1)^n = \left(\overset{x}{(10)} + \overset{y}{(-1)}\right)^n$

$$\Rightarrow 9^n = \sum_{k=0}^{n} \binom{n}{k} 10^{n-k} (-1)^n = \sum_{k=0}^{n} (-1)^n \binom{n}{k} 10^{n-k}$$

Ex: What is the coefficient of $x^6 y^7$ in $(2x+3y)^{13}$?

This will be the $k=7$ term of the binomial series (go by power of y), so the term is

$$\binom{13}{7}(2x)^6(3y)^7 = \binom{13}{7} 2^6 3^7 = 240{,}185{,}088$$

3.5 - Inclusion-Exclusion Principle

When counting the total number of things from two different groups, we must be careful not to double count.

Ex: Let $H = \{1, 2, 3, \ldots, 99, 100\}$
$H_2 = \{x \in H \mid 2|x\}$ (divisible by 2)
$H_3 = \{x \in H \mid 3|x\}$ (" " 3)

How many numbers between 1 & 100 are divisible by 2 or 3? I.e., What is $|H_2 \cup H_3|$?

$|H_2| = 50$ and $|H_3| = 33$, so is $|H_2 \cup H_3| = 83$?
No, because $H_2 \cap H_3 = H_6$ is a subset of both and so was counted twice! So,

$$|H_2 \cup H_3| = |H_2| + |H_3| - |H_2 \cap H_3| = 50 + 33 - 16 = 67$$

Inclusion-Exclusion Principal

For sets A, B

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Ex: How many 4 card hands are there where all 4 are black or none of the 4 are face cards?

Sol:

A = set of hands w/ 4 black cards

B = set of hands w/ 4 non-face cards

$|A| = \binom{26}{4}$ 26 black cards in the deck

$|B| = \binom{40}{4}$ 40 non-face cards

$A \cap B$ = set of hands w/ 4 black non-face cards

$|A \cap B| = \binom{20}{4}$

So, $|A \cup B| = \binom{26}{4} + \binom{40}{4} - \binom{20}{4}$

$= 14{,}950 + 91{,}390 - 4{,}845 = 101{,}495$